Pesin set, closing lemma and shadowing lemma in C^1 non-uniformly hyperbolic systems with limit domination

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Abstract

For a C^1 diffeomorphism, we construct a new type of Pesin blocks and thus a Pesin set (Definition 1.1). All Pesin blocks have the same degree of mean hyperbolicity (Definition 3.1) and all sufficiently long orbit segments with starting points and ending points at the same block are of the same type of quasi-hyperbolicity (Definition 3.3). We introduce a concept of limit domination (Definition 1.6), which is weaker than the usual domination. For a C^1 diffeomorphism preserving an hyperbolic ergodic measure μ with limit domination, we show the existence of our Pesin set with μ full measure, and we realize a closing lemma and shadowing lemma, comparable with Katok's closing lemma and shadowing lemma in the $C^{1+\alpha}\,(0 < \alpha < 1)$ setting. This enables us to get certain properties in C^1 setting with limit domination, which are similar to ones in the classical Pesin theory in the $C^{1+\alpha}$ setting. We present one of such properties, showing that all invariant measures supported on some μ full measure set can be approximated by periodic measures(Proposition 1.12).

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1 Introduction

In the study of differential dynamics, uniform hyperbolicity played a crucial role for many years. Thus, the development of the so-called non-uniform hyperbolicity theory by Pesin, Katok and others was an important breakthrough. However, unlike the uniform hyperbolicity theory, the non-uniform hyperbolicity theory assumes that not only the differentiability of the given dynamics is of class C^1 but also that the first derivative satisfies an α-Hölder condition for some $\alpha > 0$. Thus there appears to be a gap between these two theories. It seems therefore natural to ask: Can the non-uniform hyperbolicity theory be established only under the C^1 differentiability hypothesis? The general answer to this problem is negative. Pugh pointed out in [12] that the α-Hölder condition for the first derivative is necessary in the Pesin stable manifold theorem and thus the C^1 setting is very different from the setting of $C^{1+\alpha}$. Though this is the case, it is interesting to investigate partial basic results deduced under the weaker C^1 differentiability hypothesis plus some domination condition.

Abdenur, Bonatti and Crovisier [1] realized an analog of the Pesin stable manifold theorem in C^1 non-uniformly hyperbolic systems with domination. In the present paper, we prove results similar to the Katok [6] closing lemma and the shadowing lemma in C^1 setting with limit domination (Definition 1.6), which is weaker than the usual domination. Applying the Liao [7] closing lemma we can shadow a so-called quasi-hyperbolic orbit segment (Definition 3.2) whose starting and ending points are near by a periodic orbit. To guarantee that any orbit segment in the basin of a hyperbolic measure is quasi-hyperbolic we need the limit domination condition.

More precisely, we construct a filtration of Pesin blocks and Pesin set for a C^1 non-uniformly hyperbolic diffeomorphism with limit domination. All Pesin blocks have the same degree of mean hyperbolicity (Definition 3.1) and all sufficiently long orbit segments with starting points and ending points at the same block are of the same type of quasi-hyperbolicity (Definition 3.3). These orbit segments meet the conditions of the Liao [7] closing lemma and can be traced by periodic orbits, which gives rise to the closing lemma. A shadowing lemma in this setting is obtained by applying a generalization of Liao's lemma by Gan [4], to trace the pseudo-orbit consisting of orbit segments with starting and ending points in a given Pesin block.

For applications of our shadowing and closing lemmas, we point out that many properties in $C^{1+\alpha}$ Pesin theory whose proofs are based on Katok's closing lemma and shadowing lemma remain true in the C^1 setting with limit domination. We show one such application in the present paper. Sigmund [13] in 1970 showed that invariant measures are approximated by periodic measures for C^1 uniformly hyperbolic diffeomorphisms. This approximation property had realized in $C^{1+\alpha}$ non-uniformly hyperbolic diffeomorphisms in 2003, when Hirayama [5] proved that periodic measures are dense in the set of invariant measures supported on a full measure set with respect to a hyperbolic *mixing* measure. In 2009 Liang, Liu and Sun [9] replaced the assumption of hyperbolic *mixing* measure by a more natural and weaker assumption of hyperbolic *ergodic* measure and generalized Hirayama's result. The proofs in [5, 9] are both based on Katok's closing and shadowing

lemmas of the $C^{1+\alpha}$ Pesin theory. We establish this approximation property (Proposition 1.12) by using our shadowing lemma (Theorem 1.3, 1.9) in the C^1 setting with limit domination.

Let M be a compact D-dimensional smooth Riemannian manifold and let ρ denote the distance induced by the Riemannian metric. Denote the tangent bundle of M by TM and set $T^{\#}M = \{u \in TM | \, \|u\| = 1\}$. Denote by $\mathrm{Diff}^1(M)$ the space of C^1 diffeomorphisms of M. Hereafter let $f \in \mathrm{Diff}^1(M)$. Now we start to state our results in three subsections.

1.1 Pesin blocks, Pesin set, shadowing lemma and closing lemma

In this subsection we introduce the definitions of our new Pesin blocks and set, and then state two theorems of shadowing and closing lemma. Denote the minimal norm of an invertible linear map A by $m(A) = \|A^{-1}\|^{-1}$.

Definition 1.1. *Given $K \in \mathbb{N}$, $\zeta > 0$, and for all $k \in \mathbb{Z}^+$, we define $\Lambda_k = \Lambda_k(K, \zeta)$ to be all points $x \in M$ for which there is a splitting $T_xM = E(x) \oplus F(x)$ with the invariance property $D_xf(E(x)) = E(f(x))$ and $D_xf(F(x)) = F(f(x))$ and satisfying:*

$$(a). \qquad \frac{\log \|Df^r|_{E(x)}\| + \sum_{j=0}^{l-1} \log \|Df^K|_{E(f^{jK+r}(x))}\|}{lK + r} \le -\zeta,$$
$$\forall \ l \ge k, \ \forall \ 0 \le r \le K-1;$$

$$(b). \qquad \frac{\log m(Df^r|_{F(f^{-lK-r}(x))}) + \sum_{j=-l}^{-1} \log m(Df^K|_{F(f^{jK}(x))})}{lK + r} \ge \zeta,$$
$$\forall \ l \ge k, \ \forall \ 0 \le r \le K-1;$$

$$(c). \qquad \frac{1}{kK + r} \log \frac{\|Df^{kK+r}|_{E(x)}\|}{m(Df^{kK+r}|_{F(x)})} \le -2\zeta, \ \forall \ 0 \le r \le K-1,$$

$$\text{and} \qquad \frac{1}{K} \log \frac{\|Df^K|_{E(f^l(x))}\|}{m(Df^K|_{F(f^l(x))})} \le -2\zeta, \ \forall \ l \ge kK.$$

Denote by $\Lambda = \Lambda(K, \zeta)$ the maximal $f-$invariant subset of $\bigcup_{k \ge 1} \Lambda_k$, meaning

$$\Lambda = \bigcap_{n \in \mathbb{Z}} f^n(\bigcup_{k \ge 1} \Lambda_k).$$

We call Λ a Pesin set and call $\Lambda_k (k \ge 1)$ Pesin blocks. (see Figure 1 and 2 to explain (a), (b) and (c), respectively).

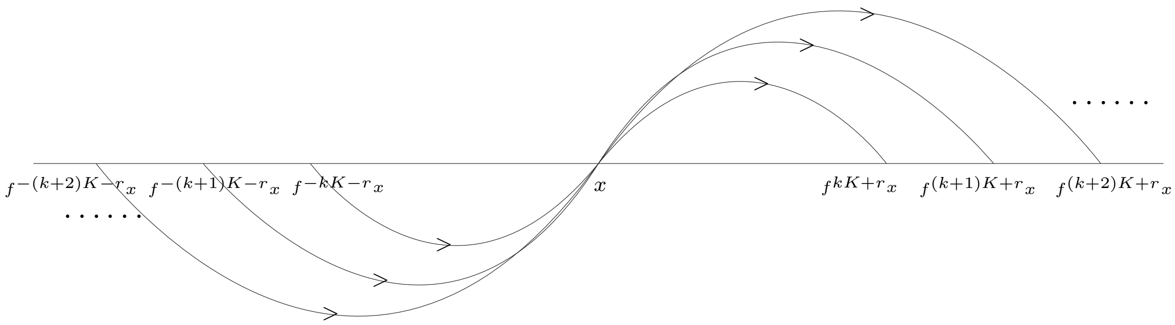

Figure 1: Graph to show (a) and (b).

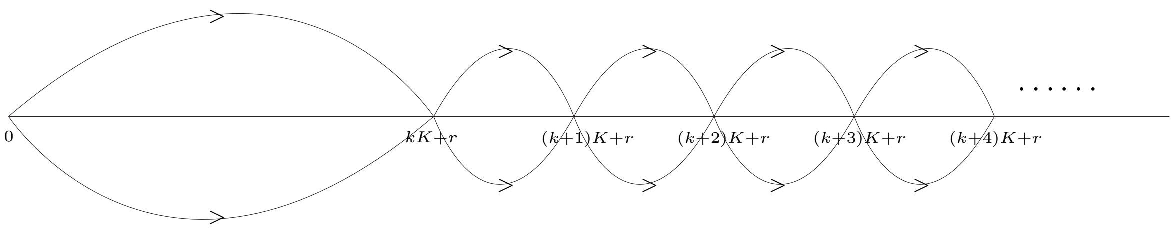

Figure 2: Graph to show (c).

Remark 1.2. For given $K \in \mathbb{N}$ and $\zeta > 0$, obviously $\Lambda_k(K, \zeta) \subseteq \Lambda_{k+1}(K, \zeta)$, $\forall\, k \in \mathbb{N}$ and $\Lambda(K, \zeta) \subseteq \Lambda(iK, \zeta)$, $\forall\, i \in \mathbb{N}$. We will illustrate our new Pesin set more precisely in Section 3.

The definition of our Pesin set is based on a generalized multiplicative ergodic theorem(Lemma 5.1) and limit domination (Definition 1.6). It enables us to realize, in Section 4 and 5, shadowing properties on nonempty Pesin blocks by using Liao's closing lemma in a C^1 non-uniformly hyperbolic system with limit domination.

To state shadowing lemma and closing lemma we need some notions. Given $x \in M$ and $n \in \mathbb{N}$, let

$$\{x,\, n\} := \{f^j(x) \mid j = 0,\, 1,\, \cdots,\, n\}.$$

In other words, $\{x, n\}$ represents the orbit segment from x. For a sequence of points $\{x_i\}_{i=-\infty}^{+\infty}$ in M and a sequence of positive integers $\{n_i\}_{i=-\infty}^{+\infty}$, we call $\{x_i,\, n_i\}_{i=-\infty}^{+\infty}$ a δ-pseudo-orbit, if $\rho(f^{n_i}(x_i),\, x_{i+1}) < \delta$ for all i. Given $\varepsilon > 0$, we call a point $x \in M$ an ε-shadowing point for a pseudo-orbit $\big\{x_i,\, n_i\big\}_{i=-\infty}^{+\infty}$, if $\rho\big(f^{c_i+j}(x), f^j(x_i)\big) < \varepsilon$, $\forall\, j = 0,\, 1,\, 2,\, \cdots,\, n_i$ and $\forall\, i \in \mathbb{Z}$, where c_i is defined as

$$c_i = \begin{cases} 0, & \text{for } i = 0 \\ \sum_{j=0}^{i-1} n_j, & \text{for } i > 0 \\ -\sum_{j=i}^{-1} n_j, & \text{for } i < 0. \end{cases} \tag{1.1}$$

Now we state shadowing lemma and closing lemma on nonempty Pesin blocks(Nonempty discussion will appear in next subsection).

Theorem 1.3. *(Shadowing lemma) If $\Lambda_k(K, \zeta) \neq \emptyset$ for some k, $K \in \mathbb{N}$ and $\zeta > 0$, then $\Lambda_k(K, \zeta)$ satisfies the following shadowing property. For $\forall\, \varepsilon > 0$, there exists $\delta > 0$ such that if a δ-pseudo-orbit $\{x_i,\, n_i\}_{i=-\infty}^{+\infty}$ satisfies $n_i \geq 2kK$ and $x_i, f^{n_i}(x_i) \in \Lambda_k(K, \zeta)$ for all i, then there exists a ε-shadowing point $x \in M$ for $\{x_i, n_i\}_{i=-\infty}^{+\infty}$. If further $\{x_i, n_i\}_{i=-\infty}^{+\infty}$ is periodic, i.e., there exists an $m > 0$ such that $x_{i+m} = x_i$ and $n_{i+m} = n_i$ for all i, then the shadowing point x can be chosen to be periodic.*

Taking $m = 1$ from Theorem 1.3, one deduces the closing lemma.

Theorem 1.4. *(Closing lemma) If $\Lambda_k(K, \zeta) \neq \emptyset$ for some k, $K \in \mathbb{N}$ and $\zeta > 0$, then $\Lambda_k(K, \zeta)$ satisfies closing property in the following sense. For $\forall \varepsilon > 0$, there exists $\delta > 0$ such that if for an orbit segment $\{x, n\}$ with length $n \geq 2kK$, one has $x, f^n(x) \in \Lambda_k(K, \zeta)$ and $\rho(x, f^n(x)) < \delta$, then there exists a point $z = z(x) \in M$ satisfying:*
(1) $f^n(z) = z$;
(2) $\rho(f^i(x),\ f^i(z)) < \varepsilon$, $i = 0,\, 1,\, \cdots, n-1$.

1.2 Existence of Pesin set for hyperbolic measures with limit domination

What systems does our Pesin set exist in? In this subsection we answer this question in C^1 non-uniformly hyperbolic systems with limit domination(defined below). Before that we introduce some notions.

We begin with the notion of ergodic measure. A Borel measure μ is said to be $f-$invariant, if $\mu(f^{-1}(B)) = \mu(B)$ for all measurable sets B. An $f-$invariant measure μ is said to be f ergodic, if the only measurable sets B with $f^{-1}(B) = B$ satisfies $\mu(B) = 0$ or $\mu(B) = 1$. Given an ergodic measure μ, by the Oseledec theorem [10], there exist
(a) real numbers, called Lyapunov exponents of μ,

$$\lambda_1 < \lambda_2 < \cdots < \lambda_W \ (W \leq D) \tag{1.2}$$

(b) positive integers $m_1, \ \cdots, \ m_R, \ \cdots, \ m_W$, satisfying $m_1 + \cdots + m_W = D$;
(c) a Borel set $L(\mu)$, called Oseledec basin of μ, satisfying $fL(\mu) = L(\mu)$, and $\mu(L(\mu)) = 1$;
(d) a measurable splitting $T_xM = E_x^1 \oplus \cdots \oplus E_x^W$ with $\dim E_x^i = m_i$ and $Df(E_x^i) = E_{f(x)}^i$, such that

$$\lim_{n\to\pm\infty} \frac{\log \|Df^n v\|}{n} = \lambda_i,$$

with uniform convergence on $\{v \in E_x^i \mid \|v\| = 1\}, \ i = 1, 2, \cdots, W$.

Now we recall the notion of hyperbolic ergodic measure.

Definition 1.5. *We call an f ergodic measure μ to be hyperbolic, if*
(1) none of the Lyapunov exponents for μ are zero;
(2) there exist Lyapunov exponents with different signs.

If μ is an f hyperbolic ergodic measure, then from Oseledec theorem above there is $R \in \mathbb{N}$ such that $\lambda_R < 0 < \lambda_{R+1}$ in the inequality (1.2). Let

$$E^s(x) := E_x^1 \oplus \cdots \oplus E_x^R$$

and

$$E^u(x) := E_x^{R+1} \oplus \cdots \oplus E_x^W,$$

then we get a measurable $Df-$invariant splitting $T_xM = E^s(x) \oplus E^u(x)$ on $L(\mu)$, called *Oseledec's hyperbolic splitting* of μ.

Before introducing limit domination, we recall the usual domination. Let Δ be an $f-$invariant set and $T_\Delta M = E \oplus F$ be a $Df-$invariant splitting on Δ. $T_\Delta M = E \oplus F$ is called (S, λ)-dominated on Δ (or simply dominated), if there exist two constants $S \in \mathbb{Z}^+$ and $\lambda > 0$ such that

$$\log \frac{\|Df^S|_{E(x)}\|}{m(Df^S|_{F(x)})} \leq -2\lambda, \ \forall x \in \Delta.$$

By Bochi and Viana [3] for C^1 generic volume preserving and non-uniformly hyperbolic symplectic systems, its Oseledec splittings are dominated. Now we start to introduce the notion of limit domination.

Definition 1.6. *Let Δ be an $f-$invariant set and $T_\Delta M = E \oplus F$ be a $Df-$invariant splitting on Δ. $T_\Delta M = E \oplus F$ is (S, λ)-limit-dominated (or simply limit-dominated), if there exist two constants $S \in \mathbb{Z}^+$ and $\lambda > 0$ such that*

$$\overline{\lim_{l\to+\infty}} \log \frac{\|Df^S|_{E(f^{lS}(x))}\|}{m(Df^S|_{F(f^{lS}(x))})} \leq -2\lambda, \ \forall x \in \Delta.$$

In this case we say that subbundles $E = \bigcup_{x\in\Delta} E(x)$ and $F = \bigcup_{x\in\Delta} F(x)$ satisfy (S, λ)-limit domination. The left limit is called limit-dominated function.

Remark 1.7. In Definition 1.6 the set Δ is not necessarily compact, and the splitting $T_xM = E(x) \oplus F(x)$ is not necessarily continuous on Δ. Since Δ is $f-$invariant, one has

$$\overline{\lim_{l\to+\infty}} \log \frac{\|Df^S|_{E(f^{lS}(x))}\|}{m(Df^S|_{F(f^{lS}(x))})} \leq -2\lambda, \, \forall\, x \in \Delta$$

$$\Leftrightarrow \overline{\lim_{l\to+\infty}} \log \frac{\|Df^S|_{E(f^{l}(x))}\|}{m(Df^S|_{F(f^{l}(x))})} \leq -2\lambda, \, \forall\, x \in \Delta.$$

Remark 1.8. On one hand, in the probabilistic perspective, limit domination is equivalent to domination. More precisely, for an invariant measure, it is easy to see that limit domination on a full measure set implies domination on all recurrent points (A point x is recurrent if there is $n_i \uparrow +\infty$ such that $\lim_{i\to+\infty} f^{n_i}(x) = x$). Since the set of recurrent points is of full measure and dominated splitting can be always extended on neighborhoods, one has domination on the support of this measure. Thus, limit domination on a full measure set implies domination on the support of this measure. On the other hand, limit domination is weaker in the geometric perspective, see Example 2.3. Furthermore, the limit-dominated function is more convenient to connect Lyapunov exponents of Birhorff average(Proposition 2.5). So we introduce limit domination in the present paper. More precise discussion will appear in Section 2.

Now we state a theorem to show the existence of Pesin set with full measure in non-uniformly hyperbolic systems with limit domination.

Theorem 1.9. *Let $f \in \mathrm{Diff}^1(M)$ preserve an hyperbolic ergodic measure μ. Assume that the Oseledec's hyperbolic splitting $T_xM = E^s(x) \oplus E^u(x)$ on $L(\mu)$ (or on a full-measured set) is (S, λ)-limit-dominated. Let λ_s and λ_u be the maximal and minimal Lyapunov exponents in E^s and E^u, respectively. Let $\beta = \min\{-\lambda_s,\, \lambda_u,\, \frac{\lambda}{S}\}$ and take ζ with $0 < \zeta < \beta$. There exists $K_0 \in \mathbb{Z}^+$ such that for all $K \geq K_0$ Pesin set $\Lambda(K, \zeta)$ is of μ full measure.*

Remark 1.10. For a non-uniformly hyperbolic system on a *surface* with $(1, \lambda)$-limit domination, since the subbudles are both one dimensional, one can take $K_0 = 1$ and get a Pesin set $\Lambda(K, \zeta)$ of full measure for any $K \geq 1$ according to the proof of Theorem 1.9 in Section 5.

1.3 Density of periodic points and measures

Denote by $Per(f)$ the set of all periodic points and denote by $\overline{Per(f)}$ the closure of $Per(f)$. As an application of Theorem 1.9 and Theorem 1.4, we get a corollary as follows.

Corollary 1.11. *Under the assumptions of Theorem 1.9, it follows that supp$(\mu) \subseteq \overline{Per(f)}$.*

The same result in the $C^{1+\alpha}$ case was first proved by Katok[6]. We omit the proof of Corollary 1.11 because it is similar to the proof in [6].

We call a measure μ to be a periodic measure, if there is a periodic point z with period p such that $\mu = \frac{1}{p}\sum_{i=0}^{p-1}\delta_{f^i(z)}$, where δ_x denotes the Dirac measure at x. Given a Pesin set $\Lambda(K,\,\zeta)$, let

$$\tilde{\Lambda}_k(K,\,\zeta) := \operatorname{supp}(\mu|_{\Lambda_k(K,\,\zeta)}) \;\text{ and }\; \tilde{\Lambda}(K,\,\zeta) := \bigcup_{k=1}^{\infty}\tilde{\Lambda}_k(K,\,\zeta).$$

Then as an application of Theorem 1.9 and 1.3, we get the following.

Proposition 1.12. *Let us make the same assumptions as in Theorem 1.9. Then there exist $K \in \mathbb{N}$ and $\zeta > 0$ such that $\tilde{\Lambda}(K,\,\zeta)$ is of μ full measure, and the set of periodic measures is dense in the set of all $f-$invariant measures supported on $\tilde{\Lambda}(K,\,\zeta)$.*

The proofs of Theorem 1.3 and 1.4 are based on the closing lemma by Liao [7] and its generalized shadowing lemma by Gan [4], which help us properly realize the shadowing and closing lemma on our new Pesin blocks. The proof of Theorem 1.9 is based on two types of weak hyperbolicity. One comes from nonzero Lyapunov exponents and the other from limit domination. These tools help us to construct a new Pesin set with full measure. The proof of Proposition 1.12 is an application of Theorem 1.9 and Theorem 1.3.

2 Limit domination

In this section we mainly discuss certain properties of limit domination itself, and the relations of limit domination and domination in topological perspective. Moreover, we point out that the limit domination is closely related to the gap of *mean* expanding Lyapunov exponent on the unstable bundle and *mean* contracting Lyapunov exponent on the stable bundle.

Similar to the case of domination whose splitting is unique if one fixes the dimensions of the subbundles (see [2]), the limit-dominated splitting is also unique if one fixes the dimensions of the subbundles. For two subbundles $E(x)$ and $F(x)$, define

$$\angle(E(x), F(x)) = \inf\{\|u-v\| : \|u\| = \|v\| = 1,\, u \in E(x), v \in F(x)\}.$$

Let

$$\alpha = \max_{x\in M}\, \log\frac{\|Df_x\|}{m(Df_x)}\,.$$

Clearly $\alpha \geq 0$. The following proposition points out more properties that limit domination has in common with domination.

Proposition 2.1. *Given two constants $S \in \mathbb{Z}^+,\; \lambda > 0$ and an f invariant set Δ, if there is an (S,λ)-limit-dominated Df-invariant splitting $T_xM = E(x) \oplus F(x)$ on Δ, then the following properties hold.*

(1) For any integers $0 \le q \le S-1$ and $k > \frac{q\alpha}{2\lambda}$, the splitting $T_xM = E(x) \oplus F(x)$ is $(kS+q, k\lambda - \frac{q\alpha}{2})$ -limit-dominated on Δ.

(2) There exists $e_0 > 0$ such that $\underline{\lim}_{n\to+\infty} \angle(E(f^{nS}(x)), F(f^{nS}(x))) \ge e_0, \ \forall x \in \Delta$.

Proof (1) Let $K = kS+q$. Since $T_xM = E(x) \oplus F(x)$ is (S,λ)-limit-dominated and

$$\frac{\|Df^K|_{E(f^l(x))}\|}{m(Df^K|_{F(f^l(x))})} \le (\prod_{i=0}^{k-1} \frac{\|Df^S|_{E(f^{l+iS}(x))}\|}{m(Df^S|_{F(f^{l+iS}(x))})}) \times \frac{\|Df^q|_{E(f^{l+kS}(x))}\|}{m(Df^q|_{F(f^{l+kS}(x))})}$$

$$\le (\prod_{i=0}^{k-1} \frac{\|Df^S|_{E(f^{l+iS}(x))}\|}{m(Df^S|_{F(f^{l+iS}(x))})}) \times e^{q\alpha},$$

by Remark 1.7 one has

$$\overline{\lim_{l\to+\infty}} \log \frac{\|Df^K|_{E(f^l(x))}\|}{m(Df^K|_{F(f^l(x))})} \le \overline{\lim_{l\to+\infty}} \sum_{i=0}^{k-1} \log \frac{\|Df^S|_{E(f^{l+iS}(x))}\|}{m(Df^S|_{F(f^{l+iS}(x))})} + q\alpha$$

$$\le -2k\lambda + q\alpha, \ \ \forall x \in \Delta.$$

Thus $T_xM = E(x) \oplus F(x)$ is $(kS+q, k\lambda - \frac{q\alpha}{2})$-limit-dominated on Δ.

(2) Let

$$c = \min_{x\in M} m(D_xf^S), \quad C = \max_{x\in M} \|D_xf^S\|.$$

Clearly c, $C \in (0,+\infty)$. By continuity of the tangent bundle T_xM, there exists real number $e_0 > 0$ such that if $\|u-v\| < e_0$, $\|u\| = \|v\| = 1$, u, $v \in T_xM$ then

$$\big|\|D_xf^S(u)\| - \|D_xf^S(v)\|\big| < c(1-e^{-\lambda}), \forall x \in M,$$

which implies

$$\frac{\|D_xf^S(u)\|}{\|D_xf^S(v)\|} \ge \frac{\|D_xf^S(v)\| - c(1-e^{-\lambda})}{\|D_xf^S(v)\|} \ge e^{-\lambda}. \tag{2.3}$$

Since $T_xM = E(x) \oplus F(x)$ is (S,λ)-limit-dominated, for any $x \in \Delta$, there exists an integer $N(x) \ge 1$ such that for $n \ge N(x)$,

$$\log \frac{\|Df^S|_{E(f^{nS}(x))}\|}{m(Df^S|_{F(f^{nS}(x))})} < -2\lambda + \lambda = -\lambda. \tag{2.4}$$

For $n \ge N(x)$ and two vectors $u \in E(f^{nS}(x)), v \in F(f^{nS}(x))$ with $\|u\| = \|v\| = 1$, we claim that $\|u-v\| \ge e_0$. Otherwise, it holds that $\|u-v\| < e_0$, and thus by the inequality (2.3),

$$\frac{\|Df^S|_{E(f^{nS}(x))}\|}{m(Df^S|_{F(f^{nS}(x))})} \ge \frac{\|D_xf^S(u)\|}{\|D_xf^S(v)\|} \ge e^{-\lambda},$$

which contradicts (2.4). So, we have

$$\angle(E(f^{nS}(x)), F(f^{nS}(x)))$$

$$= \inf\{\|u-v\| : \|u\| = \|v\| = 1,\ u \in E(f^{nS}x), v \in F(f^{nS}x)\} \geq e_0,$$

for all $n \geq N(x)$ and therefore

$$\varliminf_{n\to+\infty} \angle(E(f^{nS}(x)), F(f^{nS}(x))) \geq e_0,\ \ x \in \Delta.$$

$\square$

Clearly, (S,λ)-domination implies (S,λ)-limit domination and the later is weaker. The following proposition focuses on the inverse implication.

Proposition 2.2. *Let us make the same assumptions as in Proposition 2.1. If further Δ is compact and $T_xM = E(x) \oplus F(x)$ is continuous on Δ, then there exists $n_0 > 0$ such that for every $n \geq n_0$, $T_xM = E(x) \oplus F(x)$ is (nS, λ) dominated on Δ.*

Proof Since $T_xM = E(x) \oplus F(x)$ is (S,λ)-limit-dominated, for any $\delta > 0$, there exists an integer $l(x) \geq 1$ such that for all $l \geq l(x)$,

$$\log \frac{\|Df^S|_{E(f^{lS}(x))}\|}{m(Df^S|_{F(f^{lS}(x))})} \leq -2\lambda + \delta,\ \ x \in \Delta,$$

which implies

$$\frac{1}{l-l(x)} \sum_{i=l(x)}^{l-1} \log \frac{\|Df^S|_{E(f^{iS}(x))}\|}{m(Df^S|_{F(f^{iS}(x))})} \leq -2\lambda + \delta,\ \ \forall\ l > l(x).$$

Letting $l \to +\infty$ we have

$$\varlimsup_{l\to+\infty} \frac{1}{l} \sum_{i=0}^{l-1} \log \frac{\|Df^S|_{E(f^{iS}(x))}\|}{m(Df^S|_{F(f^{iS}(x))})}$$

$$= \varlimsup_{l\to+\infty} \frac{1}{l-l(x)} \sum_{i=l(x)}^{l-1} \log \frac{\|Df^S|_{E(f^{iS}(x))}\|}{m(Df^S|_{F(f^{iS}(x))})} \leq -2\lambda + \delta.$$

Letting $\delta \to 0$ one has

$$\varlimsup_{l\to+\infty} \frac{1}{l} \sum_{i=0}^{l-1} \log \frac{\|Df^S|_{E(f^{iS}(x))}\|}{m(Df^S|_{F(f^{iS}(x))})} \leq -2\lambda.$$

Thus we can take $n(x) \geq 1$ such that

$$\frac{1}{n(x)} \sum_{i=0}^{n(x)-1} \log \frac{\|Df^S|_{E(f^{iS}(x))}\|}{m(Df^S|_{F(f^{iS}(x))})} < -2\lambda + \lambda = -\lambda,\ \ x \in \Delta.$$

Since $TM = E \oplus F$ is continuous on Δ, there exists a neighborhood V_x of x such that for every $y \in V_x$ one has

$$\frac{1}{n(x)} \sum_{i=0}^{n(x)-1} \log \frac{\|Df^S|_{E(f^{iS}(y))}\|}{m(Df^S|_{F(f^{iS}(y))})} < -\lambda.$$

We take a finite cover $\{V_{x_1}, ..., V_{x_q}\}$ for the compact Δ and let $N = \max\{n(x_1)\,,\, ...\,,\, n(x_q)\}$. Let

$$\gamma = max_{x\in\Delta}|\log\frac{\|Df^S|_{E(x)}\|}{m(Df^S|_{F(x)})}|.$$

Then $\gamma < \infty$ because of the continuity of splittings E and F and the compactness of Δ. We define inductively a sequence $N_k : \Delta \to \mathbb{N}$ by

$$N_0(x) = 0,\ N_1(x) = min\{n(x_i) : x \in V_{x_i}, i = 1, ..., q\},$$

$$N_{k+1}(x) = N_k(x) + N_1(f^{N_k(x)S}(x)),\ k \geq 1.$$

Thus, for all $x \in \Delta$ and n, there exists k such that $N_k(x) \leq n < N_{k+1}(x)$. Hence

$$\sum_{i=0}^{n-1}\log\frac{\|Df^S|_{E(f^{iS}(x))}\|}{m(Df^S|_{F(f^{iS}(x))})} < -N_k(x)\lambda + (n - N_k(x))\gamma \leq -n\lambda + N(\lambda+\gamma).$$

By taking $n_0 = [2 + \frac{N(\lambda+\gamma)}{\lambda}] + 1$ where $[a]$ denotes the maximal integer less than or equal to a, we then have

$$\log\frac{\|Df^{nS}|_{E(x)}\|}{m(Df^{nS}|_{F(x)})} \leq \sum_{i=0}^{n-1}\log\frac{\|Df^S|_{E(f^{iS}(x))}\|}{m(Df^S|_{F(f^{iS}(x))})} \leq -2\lambda$$

for all $x \in \Delta$ and $n \geq n_0$, which deduces that $T_xM = E(x) \oplus F(x)$ is $(nS, \lambda)-$dominated on Δ. $\square$

Given an ergodic measure μ of f, if $\Delta := L(\mu)$ is compact and the stable and unstable subbundles in the Oseledec splitting are continuous, (S, λ)-limit dominated implies (nS, λ)-domination for some large integer n, which is weaker than $(S, \lambda)-$domination. However, in general, the Oseledec Basin $\Delta := L(\mu)$ is not compact, the (nS, λ)-domination cannot be derived from (S, λ)-limit domination on Δ, even if the stable and unstable subbundles in the Oseledec splitting are continuous (see Example 2.3). Therefore, $(S, \lambda)-$limit domination is indeed weaker than the usual $(S, \lambda)-$domination. In the present paper the stable and unstable subbundles in the Oseledec splitting are not necessarily continuous when one assumes the limit domination in the main theorems, thus our limit domination is strictly weaker than the usual domination.

To further illustrate the differences between limit domination and domination, we construct a simple example as follows.

Example 2.3. *Let g be a $C^r(r \geq 1)$ increasing function on $[0, 1]$, satisfying:*

$$g(0) = 0,\ g'(0) = \frac{1}{2},\ g(1) = 1,\ g'(1) = \frac{1}{2},\ g(\frac{1}{2}) = \frac{1}{2},\ g'(\frac{1}{2}) = \frac{3+\sqrt{5}}{2},\ \textit{and}$$

$$g(x) < x,\ \textit{for all}\ x \in (0, \frac{1}{2}),\ g(x) > x,\ \textit{for all}\ x \in (\frac{1}{2}, 1).$$

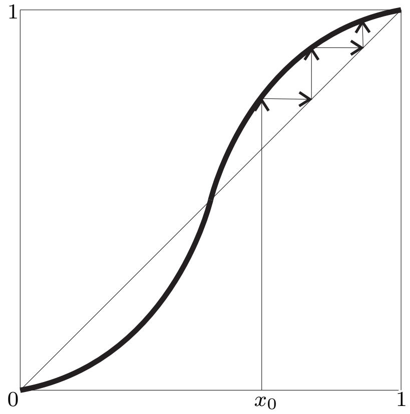

Figure 3: Graph of the function g.

And let $h : T^2 \to T^2$ *be the hyperbolic Torus automorphism*

$$(y, z) \mapsto (2y + z, y + z), \ y, z \in S^1 = \mathbb{R}/\mathbb{Z}.$$

Define $f = g \times h : T^3 \to T^3$. *Clearly,*

$$Df(x, y, z) = \begin{pmatrix} g'(x) & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

There exists naturally a continuous splitting $TT^3 = E_1 \oplus E_2 \oplus E_3$*, where* E_2 *and* E_3 *are from the hyperbolic Torus automorphism* h *and* E_1 *is* $g-$*invariant. The forward Lyapunov exponent of* E_1 *is* $\log \frac{1}{2}$ *over* $T^3 - \{\frac{1}{2}\} \times T^2$ *(we only use the forward Lyapunov exponent for the stable subbundle in construction of our new Pesin set in section 3) and the Lyapunov exponents of* $E_2 \oplus E_3$ *over* $\{0\} \times T^2$ *are* $\log \frac{3-\sqrt{5}}{2}$, $\log \frac{3+\sqrt{5}}{2}$ *respectively. Set* $E^s = E_1 \oplus E_2$ *and* $E^u = E_3$*, then* $E^s \oplus E^u$ *construct a continuous* Df*-invariant splitting of* TT^3 *over the whole space* T^3 *and the two subbundles satisfy* $(1, \log(3 + \sqrt{5})^{\frac{1}{2}})$*-limit dominated over* $T^3 - \{\frac{1}{2}\} \times T^2$ *(hint:* $\log \frac{\frac{1}{2}}{\frac{3+\sqrt{5}}{2}} = -2\log(3 + \sqrt{5})^{\frac{1}{2}}$*). However, the limit-dominated property can not be extended to the whole space* T^3 *because the splitting over every point* $(\frac{1}{2}, y, z)$ $(y, z \in S^1)$ *does not have limit domination(hint:* $\log \frac{\frac{3+\sqrt{5}}{2}}{\frac{3+\sqrt{5}}{2}} = 0$*), even though the splitting is continuous on the whole space* T^3*. In this example, the maximal* $f-$*invariant set admitting domination is* $\{0\} \times T^2$*. Moreover, every point* $(0, p)$ *is hyperbolic periodic point where* p *is a hyperbolic periodic point for* h*. Denote by* δ_0 *the point measure at point* $0 \in S^1$ *and denote by* m *the Lebesgue measure on* T^2*, then the product measure* $\mu = \delta_0 \times m$ *is a hyperbolic ergodic measure of the diffeomorphism* $f = g \times h$ *with three nonzero Lyapunov exponents* $-\log 2$, $-\log \frac{3+\sqrt{5}}{2}$, $\log \frac{3+\sqrt{5}}{2}$. $\square$

Remark 2.4. In Example 2.3 the Oseledec basin of hyperbolic ergodic measure $\mu = \delta_0 \times m$ is $L(\mu) = \mathrm{supp}(\mu) = \{0\} \times T^2$. Taking $K = 1$ and $0 < \zeta < \log 2$, our Pesin set $\Lambda(1, \zeta)$ is

$T^3 \setminus \{\frac{1}{2}\} \times T^2$ and every Pesin block $\Lambda_k(1,\zeta)$ is $[0,a_k]\bigcup[b_k,1] \times T^2$ for some $a_k \in (0,\frac{1}{2})$ and $b_k \in (\frac{1}{2},1)$. Clearly $\Lambda(K,\zeta) - (L(\mu)\bigcup \mathrm{supp}(\mu))$ is not empty and has Lebesgue full measure.

Given an integer $K > 0$ and an f^K-invariant hyperbolic measure ν, by the Oseledec theorem and the Birkhoff theorem, there exists an f^K-invariant set $B(\nu)$ with ν-full measure and a measurable Df^K invariant splitting $T_xM = E^s(x) \oplus E^u(x)$ on $B(\nu)$ such that the following limits exist

$$\lim_{l\to+\infty} \frac{1}{l}\log \|Df^{lK}|_{E^s(x)}\|, \quad \lim_{l\to+\infty} \frac{1}{l}\log m(Df^{lK}|_{E^u(x)}),$$

and

$$\lim_{l\to+\infty} \frac{1}{l}\sum_{j=0}^{l-1} \log \|Df^{K}|_{E^s(f^{jK}(x))}\|, \quad \lim_{l\to+\infty} \frac{1}{l}\sum_{j=0}^{l-1} \log m(Df^{K}|_{E^u(f^{jK}(x))}),$$

which are denoted respectively by $\lambda_s(x)$, $\lambda_u(x)$ and $\lambda^s(x)$, $\lambda^u(x)$ for every $x \in B(\nu)$. Clearly, $\lambda_s(x) \le \lambda^s(x)$ and $\lambda_u(x) \ge \lambda^u(x)$. If $\lambda^s(x) < 0$ and $\lambda^u(x) > 0$ $\nu - a.e.\ x$, we say, for convenience, that E is mean contracting, E^u is mean expanding and (f^K,ν) is mean hyperbolic. We denote

$$\lambda(x) := \overline{\lim_{l\to+\infty}} \log \frac{\|Df^K|_{E^s(f^{lK}(x))}\|}{m(Df^K|_{E^u(f^{lK}(x))})}, \ \forall\, x \in B(\nu).$$

Clearly the functions $\lambda^s(x)$, $\lambda^u(x)$ and $\lambda(x)$, $x \in B(\nu)$ are f^K-invariant. The concept of limit domination relates in a more natural way to mean expansion and mean contraction by the following proposition.

Proposition 2.5. *(1) For every $x \in B(\nu)$, $\lambda(x) \ge \lambda^s(x) - \lambda^u(x) \ge \lambda_s(x) - \lambda_u(x)$.*
(2) If ν is f^K-ergodic, then $\lambda(x), \lambda^s(x), \lambda^u(x)$ are constants ν a.e. $x \in B(\nu)$.
(3) For $x \in B(\nu)$, if the limit

$$\lim_{l\to+\infty} \log \frac{\|Df^K|_{E^s(f^{lK}(x))}\|}{m(Df^K|_{E^u(f^{lK}(x))})}$$

exists, then $\lambda(x) = \lambda^s(x) - \lambda^u(x)$.

Remark 2.6. Let us take an hyperbolic ergodic measure μ (not necessarily f^K-ergodic) together with its hyperbolic splitting $T_\Delta M = E^s \oplus F^u$ on the Oseledec basin Δ and explain the limit domination condition. Once the limit $\lambda(x) := \lim_{l\to+\infty} \log \frac{\|Df^S|_{E(f^{lS}(x))}\|}{m(Df^S|_{F(f^{lS}(x))})}$ exists, by Proposition 2.5(3) it coincides with the difference between Birkhoff averages

$$\lambda^s(x) := \lim_{l\to+\infty} \frac{1}{l}\sum_{j=0}^{l-1} \log \|Df^K|_{E^s(f^{jK}(x))}\| \quad \text{and}$$

$$\lambda^u(x) := \lim_{l\to+\infty} \frac{1}{l}\sum_{j=0}^{l-1} \log m(Df^K|_{E^u(f^{jK}(x))}),$$

which exist for $\mu - a.\,e.$ $x \in \Delta$. If further $\lambda^s(x) < 0$, $\lambda^u(x) > 0$, then $\lambda(x) = \lambda^s(x) - \lambda^u(x) < 0$. In the particular case when $dim(E^s) = dim(E^u) = 1$, the limit $\lambda(x)$ coincides with the difference between negative Lyapunov exponent

$$\lim_{l\to+\infty} \frac{1}{l} \log \|Df^{lK}|_{E^s(x)}\|$$

and the positive Lyapunov exponent

$$\lim_{l\to+\infty} \frac{1}{l} \log m(Df^{lK}|_{E^u(x)}),$$

$\mu - a.\,e.$ $x \in \Delta$. In the general case without assumption of the existence of the limit $\lambda(x)$, the upper limit

$$\overline{\lim_{l\to+\infty}} \log \frac{\|Df^S|_{E^s(f^{lS}(x))}\|}{m(Df^S|_{E^u(f^{lS}(x))})}$$

could by Proposition 2.5(1) exceed the difference $\lambda^s(x) - \lambda^u(x)$ and thus exceed the difference between the largest negative Lyapunov exponent and the smallest positive Lyapunov exponent of μ and could reach zero or even bigger. In other words, the difference between the largest negative Lyapunov exponent and the smallest positive Lyapunov exponent of μ is not enough to control the "bad situation" where Df^S expands on the "contracting" bundle $E^s(f^{lS}(x))$ more than on the "expanding" bundle $E^u(f^{lS}(x))$ on $Orb(x, f^S) := \{f^{lS}(x)\}$. Our (S, λ)-limit-domination condition requires that Df^S expands on the "contracting" bundle $E^s(f^{lS}(x))$ eventually $\lambda-$less than that on the "expanding" bundle $E^u(f^{lS}(x))$ on $Orb(x, f^S)$ for $\mu - a.\,e.$ $x \in \Delta$. In the present paper, however, without assumption of the existence of the limit

$$\lim_{l\to+\infty} \log \frac{\|Df^K|_{E^s(f^{lK}(x))}\|}{m(Df^K|_{E^u(f^{lK}(x))})},$$

we have to face the general case of two different types of weak hyperbolicity: *mean hyperbolicity* and *limit domination.*

Proof (1) By the definitions of $\lambda^s(x)$ and $\lambda^u(x)$ one has

$$\lim_{l\to+\infty} \frac{1}{l} \sum_{i=0}^{l-1} \log \frac{\|Df^K|_{E^s(f^{iK}(x))}\|}{m(Df^K|_{E^u(f^{iK}(x))})} = \lambda^s(x) - \lambda^u(x).$$

For $\varepsilon > 0$, by definition of $\lambda(x)$ there exists a positive integer $l(x) \geq 1$ such that

$$\log \frac{\|Df^K|_{E^s(f^{lK}(x))}\|}{m(Df^K|_{E^u(f^{lK}(x))})} \leq \lambda(x) + \varepsilon, \quad \forall\ l \geq l(x),$$

which implies

$$\frac{1}{l - l(x)} \sum_{i=l(x)}^{l-1} \log \frac{\|Df^K|_{E^s(f^{iK}(x))}\|}{m(Df^K|_{E^u(f^{iK}(x))})} \leq \lambda(x) + \varepsilon, \ \ \forall\ l > l(x).$$

Letting $l \to +\infty$ we have

$$\lim_{l\to+\infty} \frac{1}{l} \sum_{i=0}^{l-1} \log \frac{\|Df^K|_{E^s(f^{iK}(x))}\|}{m(Df^K|_{E^u(f^{iK}(x))})}$$

$$= \lim_{l\to+\infty} \frac{1}{l-l(x)} \sum_{i=l(x)}^{l-1} \log \frac{\|Df^K|_{E^s(f^{iK}(x))}\|}{m(Df^K|_{E^u(f^{iK}(x))})} \leq \lambda(x) + \varepsilon.$$

Letting $\varepsilon \to 0$,

$$\lambda^s(x) - \lambda^u(x) \leq \lambda(x).$$

(2) Since $\lambda(x), \lambda^s(x), \lambda^u(x)$ are f^K-invariant and ν is f^K-ergodic, $\lambda(x), \lambda^s(x), \lambda^u(x)$ are constant functions for ν.*a.e.* $x \in B(\nu)$.

(3) The proof is similar to that in (1). $\square$

3 New Pesin set

In this section we investigate more properties of the Pesin set. Before that we introduce some concepts. We begin with a notion of *degree of mean hyperbolicity.*

Definition 3.1. *Let $K \in \mathbb{N}$, $\zeta > 0$. For a given f-invariant subset Δ, let $T_xM = E(x) \oplus F(x)$, $x \in \Delta$ be a Df-invariant splitting. We call Δ a mean non-uniformly hyperbolic set with (K, ζ)-degree, if for $\forall x \in \Delta$, one has*

$$\overline{\lim_{l\to+\infty}} \sum_{j=0}^{l-1} \frac{\log \|Df^K|_{E(f^{jK}(x))}\|}{lK} \leq -\zeta, \quad \textit{and}$$

$$\underline{\lim_{l\to+\infty}} \sum_{j=-l}^{-1} \frac{\log m(Df^K|_{F(f^{jK}(x))})}{lK} \geq \zeta.$$

We recall a notion of Liao's quasi-hyperbolic orbit segment [4, 7].

Definition 3.2. *Fix arbitrarily two constants $\zeta > 0$ and $e \in \mathbb{Z}^+$ and consider an orbit segment*

$$\{x, n\} := \{f^i(x) \mid \quad i = 0, 1, 2, \cdots, n\},$$

where $x \in M$ and $n \in \mathbb{N}$. We call $\{x, n\}$ a (ζ, e, q)-quasi-hyperbolic orbit segment with respect to a splitting

$$T_xM = E^s_x \oplus E^u_x,$$

if $\dim E^s_x = q$ and there is a partition

$$0 = t_0 < t_1 < \cdots < t_m = n \quad (m \geq 1)$$

such that $t_k - t_{k-1} \le e$ *and*

(1). $\frac{1}{t_k}\Sigma_{j=1}^{k} \log \|Df^{t_j - t_{j-1}}|_{f^{t_{j-1}}(E_x^s)}\| \le -\zeta,$

(2). $\frac{1}{t_m - t_{k-1}}\Sigma_{j=k}^{m} \log m(Df^{t_j - t_{j-1}}|_{f^{t_{j-1}}(E_x^u)}) \ge \zeta,$

(3). $\frac{1}{t_k - t_{k-1}} \log \frac{\|Df^{t_k - t_{k-1}}|_{f^{t_{k-1}}(E_x^s)}\|}{m(Df^{t_k - t_{k-1}}|_{f^{t_{k-1}}(E_x^u)})} \le -2\zeta, \ \ k = 1,\, 2,\, \cdots,\, m.$

We use the notion of quasi-hyperbolic orbit segment to introduce a concept of *type of quasi-hyperbolicity.*

Definition 3.3. *Let* $k,\, K \in \mathbb{N}$, $\zeta > 0$. *For a given subset* Δ *(neither necessarily* $f-$*invariant nor* f^K*-invariant maybe), let* $T_xM = E(x) \oplus F(x)$ $(x \in \Delta)$ *be a* Df*-invariant splitting, meaning that it is invariant on each orbit* $orb(x, f)$ *(*Δ *contains not necessarily the whole orbit* $orb(x, f)$*) for* $x \in \Delta$. *We say that* Δ *is a quasi-hyperbolic set of* $(\zeta, (k+1)K)$*-type, if any orbit segment* $\{x, n\} := \{f^i(x) | i = 0,\, 1,\, \cdots,\, n\}$ $(n \ge 2kK)$ *with starting point and ending point in* Δ *is* $(\zeta, (k+1)K)$*-quasi-hyperbolic orbit segment of* f.

Now we use the above notions to present more properties for Pesin blocks and set.

Proposition 3.4. *For given* $k,\, K \in \mathbb{N}$ *and* $\zeta > 0$, *the Pesin block* $\Lambda_k(K, \zeta)$ *is closed and is a quasi-hyperbolic set of* $(\zeta, (k+1)K)$*-type, and the splitting* $T_xM = E(x) \oplus F(x)$ *on* $\Lambda_k(K, \zeta)$ *is continuous. Further, the Pesin set* $\Lambda(K, \zeta)$ *is a mean non-uniformly hyperbolic set with* (K, ζ)*-degree.*

Proof Given an orbit segment $\{x,\, n\} := \{f^i(x) \,|\, i = 0,\, 1,\, \cdots,\, n\}$ $(n \ge 2kK)$ with starting point and ending point in $\Lambda_k(K, \zeta)$, i.e., $x,\, f^n(x) \in \Lambda_k(K, \zeta)$, we show that $\{x,\, n\}$ is $(\zeta,\, (k+1)K)$-quasi-hyperbolic orbit segment.

Write $n = lK + q$, where $l \ge 2k$ and $0 \le q \le K - 1$. Let $m = l - 2k + 2$ and

$$t_0 = 0,\ t_i = (k + i - 1)K + q,\ i = 1,\, 2,\, \cdots,\, m - 1,\ t_m = lK + q.$$

Thus we get a partition

$$t_0 < t_1 < t_2 < \cdots < t_{m-1} < t_m.$$

In the partition, the starting subinterval has length of $kK + q$, the ending subinterval has length of kK, and the rest have length of K.

Since $x \in \Lambda_k(K, \zeta)$, by (a) in Definition 1.1 while taking $r = q$ we have

$$\frac{1}{t_i}\Sigma_{j=1}^{i} \log \|Df^{t_j - t_{j-1}}|_{f^{t_{j-1}}(E(x))}\|$$

$$\le \frac{\log \|Df^q|_{E(x)}\| + \sum_{j=0}^{k+i-2} \log \|Df^K|_{E(f^{jK+q}(x))}\|}{q + (k + i - 1)K} \le -\zeta,$$

$i = 1,\, 2,\, \cdots,\, m - 1$, and

$$\frac{1}{t_m}\Sigma_{j=1}^{m} \log \|Df^{t_j - t_{j-1}}|_{f^{t_{j-1}}(E(x))}\|$$

$$\leq \frac{\log \|Df^q|_{E(x)}\| + \sum_{j=0}^{l-1} \log \|Df^K|_{E(f^{jK+q}(x))}\|}{q+lK} \leq -\zeta,$$

which gives rise to the first inequality in Definition 3.2.

Notice that

$$l-k-i+2 \geq l-k-m+2 = k, \ i = 1, 2, \cdots, m.$$

Since $f^n(x) \in \Lambda_k(K, \zeta)$, by (b) in the Definition 1.1 while taking $r = 0$ we have

$$\frac{1}{t_m - t_{i-1}} \Sigma_{j=i}^m \log m(Df^{t_j - t_{j-1}}|_{f^{t_{j-1}}(F(x))})$$

$$\geq \sum_{j=-(l-k-i+2)}^{-1} \frac{\log m(Df^K|_{F(f^{jK}(f^n(x)))})}{(l-k-i+2)K} \geq \zeta,$$

$i = 2, \cdots, m$, and while taking $r = q$ we have

$$\frac{1}{t_m - t_0} \Sigma_{j=1}^m \log m(Df^{t_j - t_{j-1}}|_{f^{t_{j-1}}(F(x))})$$

$$\geq \frac{\log m(Df^q|_{F(x)}) + \sum_{j=-l}^{-1} \log m(Df^K|_{F(f^{jK}(f^n(x)))})}{lK+q} \geq \zeta,$$

which gives rise to the second inequality in Definition 3.2.

Before continuing our proof we present Figure 4 to illustrate the concepts. We denote $f^{t_i}(x)$ by t_i and take $k = 3, \ l = 10, \ q = 1, \ m = 6$ and $n = 10K + 1$ in the Figure, and draw a graph for the inequality (1) and (2) of $(\zeta, 4K) = (\zeta, (3+1)K)$ quasi-hyperbolic orbit segment $\{x, 10K + 1\}$.

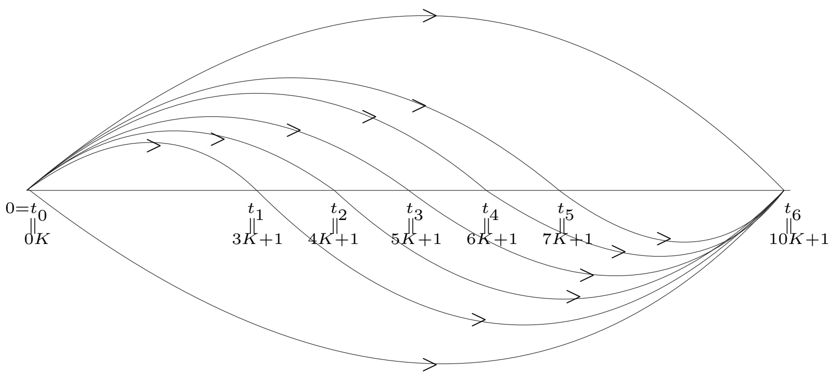

Figure 4: inequalities (1), (2) of $(\zeta, 4K)$ quasi-hyperbolic orbit segment.

Now we continue our proof and verify the third inequality in Definition 3.2. Since $x \in \Lambda_k(K, \zeta)$ and $t_1 = kK + q$, by (c) in Definition 1.1 while taking $r = q$ we have

$$\frac{1}{t_1 - t_0} \log \frac{\|Df^{t_1 - t_0}|_{f^{t_0}(E(x))}\|}{m(Df^{t_1 - t_0}|_{f^{t_0}(F(x))})}$$

$$= \frac{1}{kK+q} \log \frac{\|Df^{kK+q}|_{E(x)}\|}{m(Df^{kK+q}|_{F(x)})} \le -2\zeta,$$

while noting that $t_i \ge kK$, $i = 1, 2, \cdots, m-2$, we have

$$\frac{1}{t_i - t_{i-1}} \log \frac{\|Df^{t_i - t_{i-1}}|_{f^{t_{i-1}}(E(x))}\|}{m(Df^{t_i - t_{i-1}}|_{f^{t_{i-1}}(F(x))})}$$

$$= \frac{1}{K} \log \frac{\|Df^{K}|_{f^{t_{i-1}}(E(x))}\|}{m(Df^{K}|_{f^{t_{i-1}}(F(x))})} \le -2\zeta, \ i = 2, \cdots, m-1,$$

and while $t_m - t_{m-1} = kK$ and $t_{m-1} + jK \ge kK$, we have

$$\frac{1}{t_m - t_{m-1}} \log \frac{\|Df^{t_m - t_{m-1}}|_{f^{t_{m-1}}(E(x))}\|}{m(Df^{t_m - t_{m-1}}|_{f^{t_{m-1}}(F(x))})}$$

$$\le \frac{1}{kK} \sum_{j=0}^{k-1} \log \frac{\|Df^{K}|_{f^{t_{m-1}+jK}(E(x))}\|}{m(Df^{K}|_{f^{t_{m-1}+jK}(F(x))})} \le -2\zeta,$$

which gives rise to the third inequality in Definition 3.2.

Before continuing our proof we also present Figure 5 to give an explanation. We denote $f^{t_i}(x)$ by t_i and take $k = 3$, $l = 10$, $q = 1$, $m = 6$ and $n = 10K + 1$ in the Figure, and draw a graph for the inequality (3) of $(\zeta, 4K) = (\zeta, (3+1)K)$ quasi-hyperbolic orbit segment $\{x, 10K+1\}$.

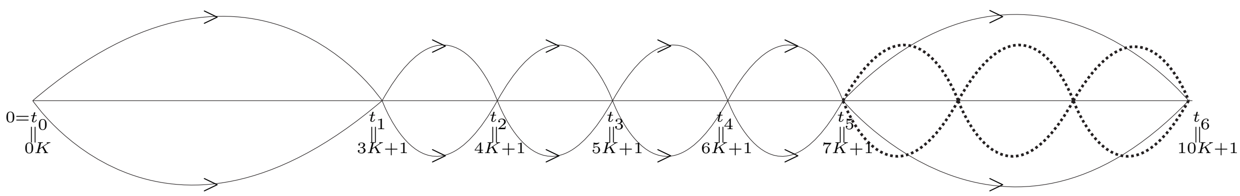

Figure 5: inequality (3) of $(\zeta, 4K)$ quasi-hyperbolic orbit segment.

Observe from the partition that $t_i - t_{i-1} \le (k+1)K$. So $\{x, n\}$ is a $(\zeta, (k+1)K)$-quasi-hyperbolic orbit segment. Thus, by Definition 3.3 $\Lambda_k(K, \zeta)$ is a quasi-hyperbolic set of $(\zeta, (k+1)K)$-type.

Next we show that $\Lambda_k(K, \zeta)$ is closed. Clearly the three conditions in the definition of $\Lambda_k(K, \zeta)$ imply that the splitting $T_xM = E(x) \oplus F(x)$ is unique. If $x \in M$ and $x_i \in \Lambda_k(K, \zeta)(i \ge 1)$ is a convergent sequence with $\lim_{i \to +\infty} x_i = x$, with the choice of k fixed, then by a compactness argument we can choose a convergent subsequence of the subspaces $\xi(x_{i_j}) \to \xi'(x)$, as $j \to +\infty$ where $\xi = E, \ F$. By assumption $x_{i_j} \in \Lambda_k(K, \zeta)$, conditions in the definition of $\Lambda_k(K, \zeta)$ are satisfied by $E(x_{i_j})$ and $F(x_{i_j})$. Letting $j \to +\infty$, then the three conditions in the definition of $\Lambda_k(K, \zeta)$ hold for the subbundles $E'(x)$ and $F'(x)$. By the uniqueness condition above, $E'(x) = E(x)$ and $F'(x) = F(x)$. So $x \in \Lambda_k(K, \zeta)$ and thus $\Lambda_k(K, \zeta)$ is closed.

By the uniqueness condition above, there is only one possible limit for $E(x_{i_j})$ and $F(x_{i_j})$. Thus the splitting $x \mapsto E(x) \oplus F(x)$ is continuous on $\Lambda_k(K, \zeta)$. That $\Lambda(K, \zeta)$ is a mean non-uniformly hyperbolic set with (K, ζ)-degree is an easy consequence.

By (a) and (b) in Definition 1.1, it is easy to see that the Pesin set $\Lambda(K, \zeta)$ is a mean non-uniformly hyperbolic set with (K, ζ)-degree. $\square$

In contrast to the situation for the sets $\Lambda_k(K, \zeta), k \geq 1$, we should observe that, in general, Λ itself need not necessarily be compact, nor is it necessarily true that the splitting $T_xM = E(x) \oplus F(x)$ is continuous on $\Lambda(K, \zeta)$. We would like to discuss more about the comparison between our Pesin set and the Pesin set in the usual sense (see for example Chapter 4 in [11]). The k-th Pesin block in the usual sense is hyperbolic with a degree corresponding to k, the degree becomes weaker as k becomes bigger. Keeping a given degree of hyperbolicity a trajectory with starting and ending points in a given Pesin block in the usual sense is, in general, not possible to stay long time outside this block. Each orbit segment contained in our $k-th$ Pesin block is mean hyperbolic of $(\zeta, (k+1)K)$ degree, where neither K nor ζ depends on k. Keeping a given degree of *mean* hyperbolicity a trajectory with starting and ending points in a fixed Pesin block in our definition can stay sufficiently long time outside this block. This allows us to trace a long trajectory with starting and ending points in a given block by periodic orbit by using Liao's lemma[7, 4] (see Lemma 4.1 below), which plays an important role in approximating every invariant measure by periodic measures in the proof of Proposition 1.12.

4 Proof of Theorem 1.3

To complete the proof of Theorem 1.3, we need Liao' closing lemma [7] and its generalization by Gan [4]. Before that we need to a concept of quasi-hyperbolic pseudo-orbit. Let $\zeta > 0,\ e,\, q \in \mathbb{Z}^+,\ \delta > 0$. Given a sequence of orbit segments $\{x_i,\, n_i\}_{i=-\infty}^{+\infty}$, we call $\{x_i,\, n_i\}_{i=-\infty}^{+\infty}$ a $(\zeta,\, e)$-quasi-hyperbolic δ-pseudo-orbit with respect to splittings $T_{x_i}M = E^s(x_i) \oplus E^u(x_i)$, if for any i, $\dim(E^s(x_i)) = q$ is independent of the choice of i, $\{x_i,\, n_i\}_{i=-\infty}^{+\infty}$ is a δ-pseudo-orbit and every orbit segment $\{x_i, n_i\}$ is a $(\zeta,\, e,\, q)$-quasi-hyperbolic orbit segment with respect to the i-th splitting $T_{x_i}M = E^s(x_i) \oplus E^u(x_i)$. Now we state Liao' closing lemma [7] and its generalization [4].

Lemma 4.1. *([4, 7]) For any $\zeta > 0$, $e \in \mathbb{Z}^+$, there exist $L > 0$, $d_0 > 0$ with following property for any $d \in (0, d_0]$. If for a $(\zeta,\, e)$-quasi-hyperbolic d-pseudo-orbit $\{x_i, n_i\}_{i=-\infty}^{+\infty}$ with respect to $Df-$invariant splittings $T_{x_i}M = E^s(x_i) \oplus E^u(x_i)$, one has $Df^{n_i}(E^\xi(x_i)) \cap T^\# M \subseteq U(E^\xi(x_{i+1}) \cap T^\# M, d)$ $(\xi = s, u)$ for all i, then there exists a Ld-shadowing point $x \in M$ for $\{x_i, n_i\}_{i=-\infty}^{+\infty}$. Moreover, if $\{x_i, n_i\}_{i=-\infty}^{+\infty}$ is periodic, i.e., there exists an $m > 0$ such that $x_{i+m} = x_i$ and $n_{i+m} = n_i$ for all i, then the shadowing point x can be chosen to be periodic with period c_m, where c_i is the same as in (1.1).*

Proof of Theorem 1.3 For $\zeta > 0$ and $e = (k+1)K$, by Lemma 4.1 there exist $L = L(k, K, \zeta) > 0$, $d_0 = d_0(k, K, \zeta) > 0$ with the shadowing property as stated in Lemma 4.1 for any $d \in (0, d_0]$.

Now we consider $d < \frac{\varepsilon}{L}$. By Proposition 3.4, Λ_k is closed, and the stable subbundle and unstable subbundle in the Oseledec splitting are continuous restricted on Λ_k. Take $\delta \in (0, d)$ such that for all $x, y \in \Lambda_k$ with $\rho(x, y) < \delta$ we have $E^\xi_x \cap T^\# M \subseteq U(E^\xi_y \cap T^\# M, d)$, $\xi = s, u$.

For a given δ-pseudo-orbit $\{x_i, n_i\}_{i=-\infty}^{+\infty}$ with $x_i, f^{n_i}(x_i) \in \Lambda_k$ and $n_i \geq 2kK(\forall i)$, we get by Proposition 3.4 that every orbit segment $\{x_i, n_i\}$ is $(\zeta,\, e)$ quasi-hyperbolic orbit segment. So $\{x_i, n_i\}_{i=-\infty}^{+\infty}$ is a $(\zeta,\, e)$ quasi-hyperbolic d-pseudo-orbit. That $\rho(f^{n_i}(x_i), x_{i+1}) < \delta$ implies that

$$Df^{n_i}(E^\xi(x_i)) \cap T^\# M \subseteq U(E^\xi(x_{i+1}) \cap T^\# M, d) \ (\xi = s, u)\text{for all } i.$$

By Lemma 4.1 there exists a Ld-shadowing point $x \in M$ for $\{x_i, n_i\}_{i=-\infty}^{+\infty}$ and thus x is also a ε-shadowing point. In particular, if $\{x_i, n_i\}_{i=-\infty}^{+\infty}$ is periodic, i.e., there exists an $m > 0$ such that $x_{i+m} = x_i$ and $n_{i+m} = n_i$ for all i, then the point x can be chosen to be periodic and thus we complete the proof. $\square$

5 Proof of Theorem 1.9

In this section we prove Theorem 1.9, and in order to do so we need a multiplicative ergodic theorem on the Birkhoff average, which approximates the Lyapunov exponent. Liao gave such a theorem in [8]. We use the multiplicative ergodic theorem by [1].

Lemma 5.1. *Let $f \in \mathrm{Diff}^1(M)$ preserve an ergodic invariant probability measure μ, and $E \subseteq TM$ be a $Df-$invariant subbundle defined over an $f-$invariant set with μ full measure. Let λ^+_E be the maximal Lyapunov exponent in E of the measure μ. Then, for any $\varepsilon > 0$, there exists an integer N_ε such that for μ almost every point $x \in M$ and any $N \geq N_\varepsilon$, the Birkhoff averages*

$$\frac{1}{lN} \sum_{j=0}^{l-1} \log \|Df^N|_{E(f^{jN}(x))}\|$$

converge towards a number contained in $[\lambda^+_E, \lambda^+_E + \varepsilon)$, when $l \to +\infty$.

The similar property holds for the minimal Lyapunov exponent.

Proof This is Lemma 8.4 in [1], where E is assumed to be continuous on the support $\mathrm{supp}(\mu)$. But the continuity of E and the compactness of $\mathrm{supp}(\mu)$ contribute nothing to their proof there, one can thus get the version in Lemma 5.1 by repeating the proof word by word. $\square$

Proof of Theorem 1.9 Take $\chi = \frac{1}{2}(\beta + \zeta)$, then $\chi < \max\{-\lambda_s, \lambda_u\}$. By Lemma 5.1 there exists $K_1 = K_1(\chi) \in \mathbb{Z}^+$ and an f-invariant subset $B(\mu) \subseteq L(\mu)$ of μ full measure such that for any $K \geq K_1$, $\forall x \in B(\mu)$, we have

$$\lim_{l\to+\infty} \sum_{j=0}^{l-1} \frac{\log \|Df^K|_{E^s(f^{jK}(f^r x))}\|}{lK} < -\chi, \quad \lim_{l\to+\infty} \sum_{j=-l}^{-1} \frac{\log m(Df^K|_{E^u(f^{jK}(x))})}{lK} > \chi,$$

which implies that for $\forall\ 0 \le r \le K-1$,

$$\lim_{l\to+\infty} \frac{\log\|Df^r|_{E^s(x)}\| + \sum_{j=0}^{l-1}\log\|Df^K|_{E^s(f^{jK+r}(x))}\|}{lK+r} < -\chi, \tag{5.5}$$

and

$$\lim_{l\to+\infty} \frac{\log m(Df^r|_{E^u(f^{-(lK+r)}(x))}) + \sum_{j=-l}^{-1}\log m(Df^K|_{E^u(f^{jK}(x))})}{lK+r} > \chi. \tag{5.6}$$

Recall $\alpha = \max_{x\in M}\log\frac{\|Df_x\|}{m(Df_x)}$. Taking

$$K_0 \ge \max\{K_1, \frac{(S-1)(2\beta+\alpha)}{\beta-\zeta}\},$$

we claim that the splitting $E^s \oplus E^u$ is $(K, \frac{K(\zeta+\beta)}{2})$-limit-dominated on $B(\mu) \subseteq L(\mu)$ for all $K \ge K_0$. In fact, let $K = nS + q$ for some $0 \le q \le S-1$. Since $T_xM = E^s(x) \oplus E^u(x)$ is (S,λ)-limit-dominated on $L(\mu)$, one has by Proposition 2.1

$$\overline{\lim_{l\to+\infty}} \log\frac{\|Df^K|_{E(f^l(x))}\|}{m(Df^K|_{F(f^l(x))})} \le -2n\lambda + q\alpha,\ \ \forall x \in B(\mu).$$

Note that $0 < \zeta < \beta \le \frac{\lambda}{S}$, $q \le S-1$ and $K \ge K_0$, we have

$$-2n\lambda + q\alpha \le -2(K-q)\beta + q\alpha \le -2K\beta + (S-1)(2\beta+\alpha) \le -K(\beta+\zeta).$$

Thus the splitting $E^s \oplus E^u$ is $(K, \frac{K(\zeta+\beta)}{2})$-limit-dominated on $B(\mu)$.

For the above $K \ge K_0$, we claim that the Pesin block $\Lambda_k(K,\zeta)$ is not empty for large k. Fix $x \in B(\mu)$, by (5.5) and (5.6) there exists $k_1 = k_1(x)$ such that for all $l \ge k_1$, $\forall\, 0 \le r \le K-1$, one has

$$\frac{\log\|Df^r|_{E^s(x)}\| + \sum_{j=0}^{l-1}\log\|Df^K|_{E^s(f^{jK+r}(x))}\|}{lK+r} \le -\zeta,$$

$$\frac{\log m(Df^r|_{E^u(f^{-(lK+r)}(x))}) + \sum_{j=-l}^{-1}\log m(Df^K|_{E^u(f^{jK}(x))})}{lK+r} \ge \zeta.$$

Since the splitting $E^s \oplus E^u$ is $(K, \frac{K(\zeta+\beta)}{2})$-limit-dominated on $B(\mu)$ and $-(\zeta+\beta) < -2\zeta$, by Remark 1.7 we can choose $k_2 = k_2(x)$ such that for all $l \ge k_2K$,

$$\frac{1}{K}\log\frac{\|Df^K|_{E^s(f^l(x))}\|}{m(Df^K|_{E^u(f^l(x))})} \le -2\zeta.$$

Using the inequality

$$\lim_{l\to+\infty}\frac{1}{lK+r}\log\frac{\|Df^{lK+r}|_{E^s(x)}\|}{m(Df^{lK+r}|_{E^u(x)})} = \lambda_s - \lambda_u \le -2\beta,$$

there exists $k_3 = k_3(x)$ such that for all $l \geq k_3$ and $r = 0, 1, \cdots, K-1$,

$$\frac{1}{lK+r} \log \frac{\|Df^{lK+r}|_{E^s(x)}\|}{m(Df^{lK+r}|_{E^u(x)})} \leq -2\zeta.$$

Take $k \geq max\{k_1, k_2, k_3\}$ and then the three conditions in Definition 1.1 hold. Hence $x \in \Lambda_k(K, \zeta) \neq \emptyset$ and the claim is true. We have then that

$$B(\mu) \subseteq \bigcup_{k \geq 1} \Lambda_k(K, \zeta).$$

Recall that the Pesin set $\Lambda(K, \zeta)$ is the maximal f-invariant set in $\bigcup_{k\geq 1} \Lambda_k(K, \zeta)$ and $B(\mu)$ is f-invariant, one has $B(\mu) \subseteq \Lambda(K, \zeta)$ and thus our Pesin set Λ is of μ full measure. $\square$

In the proof of Theorem 1.9, we make use of three points of weak hyperbolicity: the maximal forward Lyapunov exponent for the stable bundle, the minimal backward and forward Lyapunov exponents for the unstable bundle and (forward) limit domination. Let us explain more precisely. Liao's quasi-hyperbolic orbit segment allows a big time step e and thus from its definition the first interval of the partition can be chosen longer. For an hyperbolic ergodic measure in C^1 setting, Lemma 5.1 ensures the first and second conditions in the definition of our Pesin set. Since the gap of the forward maximal Lyapunov exponent on the stable bundle and the forward minimal Lyapunov exponent on the unstable bundle can deduce the needed hyperbolicity for a large time step, the third condition in the definition of our Pesin set naturally holds for the first interval of the partition. But the conditions required for remaining intervals can't be deduced. Thus limit domination is needed and enough.

6 Weak specification property and Approximation property of Invariant measures

In this section we present a lemma for a weaker specification property and prove Proposition 1.12. Before that we recall $\tilde{\Lambda}_k(K, \zeta) = \text{supp}(\mu|_{\Lambda_k(K,\zeta)})$ and $\tilde{\Lambda}(K, \zeta) = \bigcup_{k=1}^{\infty} \tilde{\Lambda}_k(K, \zeta)$ for a given Pesin set $\Lambda(K, \zeta)$.

Lemma 6.1. *Let $f \in \text{Diff}^1(M)$ preserve an ergodic invariant probability measure μ. Then for every μ positive-measured set $\tilde{\Lambda}_k(K, \zeta)$, one has the weak specification property on it, that is, for any $\varepsilon > 0$, there exist $X_\tau = X_\tau(k, \varepsilon) > 0 (\tau = 1, 2)$ with the following property. If for a given sequence of points $x_1, x_2, \cdots, x_N \in \widetilde{\Lambda}_k (N \in \mathbb{N})$ and a sequence of positive numbers $n_1, n_2, \cdots, n_N$, one has $n_i \geq 2kK$ and $f^{n_i} x_i \in \widetilde{\Lambda}_k$ for $i = 1, 2, \cdots, N$, then there exist a periodic point $z \in M$, a positive number $p \in [\sum_{i=1}^{N} n_i + NX_1, \sum_{i=1}^{N} n_i + NX_2]$, and a sequence of nonnegative numbers $c_0 = 0, c_1, \cdots, c_{N-1}$, such that*
(1) $f^p z = z$;
(2) $\rho(f^{c_{i-1}+j} z, f^j x_i) < \varepsilon, \forall\, j = 0, 1, \cdots, n_i, i = 1, 2, \cdots, N$.

Proof of Lemma 6.1 Given Pesin block $\Lambda_k = \Lambda_k(K,\, \zeta)$ and $\varepsilon > 0$, by Theorem 1.3 there exists $\delta > 0$ such that for any periodic δ-pseudo-orbit $\{x_i,\, n_i\}_{i=-\infty}^{+\infty}$ with $n_i \geq 2kK$ and $x_i, f^{n_i}(x_i) \in \Lambda_k(K,\, \zeta)\ \forall\ i$, there is a ε-shadowing periodic point $x \in M$ for $\{x_i, n_i\}_{i=-\infty}^{+\infty}$.

Choose and fix for $\widetilde{\Lambda}_k$ a finite cover $\alpha = \{V_1, V_2, \cdots, V_{r_0}\}$ by nonempty open balls V_i in M such that $diam(U_i) < \delta$ and $\mu(U_i) > 0$ where $U_i = V_i \cap \widetilde{\Lambda}_k$, $i = 1,\, 2,\, \cdots,\, r_0$. Since μ is f ergodic, by Birkhoff ergodic theorem we have

$$\lim_{l\to+\infty} \frac{1}{l} \sum_{n=0}^{l-1} \mu(f^{-n}(U_i) \cap U_j) = \mu(U_i)\mu(U_j) > 0. \tag{6.7}$$

Take

$$X_{i,j} = \min\{n \in \mathbb{N} \mid n \geq 2kK,\ \mu(f^{-n}(U_i) \cap U_j) > 0\}. \tag{6.8}$$

By (6.7), $1 \leq X_{i,j} < +\infty$. Let

$$X_1 = \min_{1\leq i,j\leq r_0} X_{i,j} > 0, \quad X_2 = \max_{1\leq i,j\leq r_0} X_{i,j} > 0.$$

Now let us consider a given sequence of points $x_1,\, x_2,\, \cdots,\, x_N \in \widetilde{\Lambda}_k$, and a sequence of positive numbers $n_1, n_2, \cdots, n_N$ satisfying $n_i \geq 2kK$ and $f^{n_i}x_i \in \widetilde{\Lambda}_k$. Fix $U_{i_0}, U_{i_1} \in \alpha$ so that

$$x_i \in U_{i_0}, f^{n_i}x_i \in U_{i_1}, i = 1,\, 2,\, \cdots,\, N.$$

Take $y_i \in U_{i_1}$ by (6.8) such that $f^{X_{(i+1)_0,i_1}} y_i \in U_{(i+1)_0}$ for $i = 1,\, 2,\, \cdots,\, N-1$ and choose $y_N \in U_{N_1}$ such that $f^{X_{(N+1)_0,N_1}} y_N \in U_{1_0}$. Thus we get a periodic δ-pseudo-orbit in M:

$$\{f^t(x_1)\}_{t=0}^{n_1} \cup \{f^t(y_1)\}_{t=0}^{X_{2_0,1_1}} \cup \{f^t(x_2)\}_{t=0}^{n_2} \cup \{f^t(y_2)\}_{t=0}^{X_{3_0,2_1}}$$

$$\cup \cdots\cdots \cup \{f^t(x_N)\}_{t=0}^{n_N} \cup \{f^t(y_N)\}_{t=0}^{X_{(N+1)_0,N_1}}$$

satisfying

$$x_i,\ f^{n_i}(x_i),\ y_i,\ f^{X_{(i+1)_0,i_1}} y_i \in \widetilde{\Lambda}_k \subseteq \Lambda_k\ \ (\forall i).$$

Hence by Theorem 1.3 there exists a periodic point $z \in M$ with period $p = \sum_{i=1}^{N}(n_i + X_{(i+1)_0,i_1})$ ε-shadowing the above sequence. More precisely,

$$\rho(f^{c_{i-1}+j}z, f^j x_i) < \varepsilon,\ \forall\, j = 0,\, 1,\, \cdots,\, n_i,\ \ i = 1,\, 2,\, \cdots,\, N,$$

where

$$c_i = \begin{cases} 0, & \text{for}\ \ i = 0 \\ \sum_{j=1}^{i}[n_j + X_{(j+1)_0,j_1}], & \text{for}\ i = 1,\, 2,\, \cdots,\, N. \end{cases}$$

Clearly $p \in [\sum_{i=1}^{N} n_i + NX_1,\ \sum_{i=1}^{N} n_i + NX_2]$. $\square$

Proof of Proposition 1.12 To deduce the density property of periodic measures, the weaker specification property in Lemma 6.1 is enough. One can take similar steps as in [5] or [9]. Here we omit the details. $\square$

References.

1. F. Abdenur, C. Bonatti, S. Crovisier, *Nonuniform hyperbolicity of C^1-generic diffeomorphisms*, Israel Journal of Mathematics, to appear.
2. Bonatti, Lorenzo, Viana, *Dynamics beyond uniform hyperbolicity: a global geometric and probabilistic perspective*, Springer-Verlag Berlin Heidelberg (2005), 287-293.
3. J. Bochi, M. Viana, *The Lyapunov exponents of generic volume preserving and symplectic systems*, Ann. of Math. 161 (2005), 1423-1485.
4. S. Gan, *A generalized shadowing lemma*, Dist. Cont. Dyn. Sys. 8 (2002), 627-632.
5. M. Hirayama, *Periodic probability measures are dense in the set of invariant measures*, Dist. Cont. Dyn. Sys. 9 (2003), 1185-1192.
6. A. Katok, *Liapunov exponents, entropy and periodic orbits for diffeomorphisms*, Pub. Math. IHES, 51 (1980), 137-173.
7. S. Liao, *An existence theorem for periodic orbits*, Acta Sci. Natur. Univ. Pekin. (1979), 1-20.
8. S. Liao, *On characteristic exponents construction of a new Borel set for the multiplicative ergodic theorem for vecter fields*, Acta Sci. Natur. Univ. Pekin. (1993), 277-302.
9. C. Liang, G. Liu, W. Sun, *Approximation properties on invariant measure and Oseledec splitting in non-uniformly hyperbolic systems*, Trans. Amer. Math. Soci. 361 (2009), 1543-1579.
10. V. I. Oseledec, *Multiplicative ergodic theorem, Liapunov characteristic numbers for dynamical systems*, Trans. Moscow Math. Soc., 19 (1968), 197-221; translated from Russian.
11. M. Pollicott, *Lectures on ergodic theory and Pesin theory on compact manifolds*, Cambridge Univ. Press, (1993).
12. C. Pugh, *The $C^{1+\alpha}$ hypothesis in Pesin theory*, (English) [J] Publ. Math., Inst. Hautes tud. Sci. 59 (1984), 143-161 .
13. K. Sigmund, *Generic properties of invariant measures for Axiom A-diffeomorphisms*, Invent.Math. 11 (1970), 99-109.